

Describing Realistic Wealth Distributions with the Extended Yard-Sale Model of Asset Exchange*

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The Extended Yard-Sale Model of asset exchange is an agent-based economic model with binary transactions, and simple models of redistribution and Wealth-Attained Advantage. As recently shown [1], the model exhibits a second-order phase transition to a coexistence regime with partial wealth condensation. The evolution of its wealth distribution is described by a nonlinear, nonlocal Fokker-Planck equation. In this work, we demonstrate that solutions to this equation fit remarkably well to the actual wealth distribution of the U.S. in 2013. The two fit parameters provide evidence that the U.S. wealth distribution is partially wealth condensed.

I. INTRODUCTION

Asset-exchange models are agent-based models of economies with binary transactions, first proposed by Angle [2]. The application of kinetic theory to such models was first undertaken by Ispolatov, Krapivsky and Redner [3], who derived a Boltzmann equation for the agent density function of one such model.

The Yard-Sale Model (YSM) of asset exchange is a particular asset-exchange model, proposed by Chakraborti in 2002 [4], and described in some detail by Hayes [5] later that same year. In the basic YSM, each of N agents has a single positive scalar attribute, which we call wealth w . In each transaction, a certain amount of wealth is moved from one transacting agent to another. The amount of wealth moved is proportional to the wealth of the poorer of the two agents, with proportionality constant β . The direction in which it is moved is decided by the flip of a fair coin. A Boltzmann equation for the YSM was first given by Boghosian [6] in 2014.

In the limit of small β , known as the *small-transaction limit*, it was also demonstrated [6] that the Boltzmann equation for the YSM reduces to a nonlinear, integrodifferential Fokker-Planck equation. In 2015, Boghosian, Johnson and Marcq [7] showed that the Gini coefficient is a Lyapunov functional of both the Boltzmann and Fokker-Planck equations, and hence that the basic YSM concentrates wealth from any initial condition.

Because the complete concentration of wealth is unrealistic, some attention has been given to modifying the YSM to account for (i) redistribution, and (ii) bias of the coin based on the relative wealth of the transacting agents. A simple Ornstein-Uhlenbeck model [8] of redistribution was

introduced in [6], and studied in much greater detail by Boghosian, Devitt-Lee, Johnson, Marcq and Wang [1] in 2015. The redistribution mechanism can be thought of as a tax τ levied on all agents on a per-transaction basis, pooled and redistributed uniformly to all N agents. This stabilizes the distribution, resulting in a steady state that shares some features with the famous Pareto distribution [9], especially in the limit of low τ . For higher τ it gives rise to a Gibrat distribution [10], albeit with a gaussian cutoff at very large values of wealth.

A constant bias of the coin in favor of the poorer agent was considered by Moukarzel [11] in 2007, who showed that it could stabilize the tendency of the model to concentrate wealth, and who demonstrated the existence of a first-order phase transition between a regime of complete wealth concentration and one with a stable steady state as the bias in favor of the poorer agent is increased. Because there is substantial evidence that the actual bias in the real world favors the wealthier agent rather than the poorer agent, reference [1] also introduced a model of Wealth-Attained Advantage (WAA). In this model, the coin is biased in favor of the wealthier agent by an amount proportional to the difference in wealth between the richer and poorer agent, times a WAA coefficient, ζ . Because the bias is proportional to the wealth difference in this model, it naturally reduces to zero when the transacting agents have equal wealth. The authors went on to show that this model exhibits a second-order phase transition to a wealth-condensed [12] regime when $\zeta = \tau$. Specifically, they showed that for $\zeta > \tau$, a finite fraction $1 - \tau/\zeta$ of the population's total wealth "condenses" to the wealthiest agent – or, in the continuum limit, to the wealthiest vanishingly small fraction of an agent.

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II. MATHEMATICAL DESCRIPTION

Henceforth, we refer to the YSM with both redistribution and WAA as the Extended Yard-Sale Model (EYSM). The Fokker-Planck equation for agent density distribution $P(w, t)$ in this model is [1]

$$\begin{aligned} \frac{\partial P}{\partial t} = & -\frac{\partial}{\partial w} \left[\tau \left(\frac{W}{N} - w \right) P \right] \\ & + \frac{\partial}{\partial w} \left\{ \zeta \left[2 \frac{N}{W} \left(B - \frac{w^2}{2} A \right) + (1 - 2L) w \right] P \right\} \\ & + \frac{\partial^2}{\partial w^2} \left[\left(B + \frac{w^2}{2} A \right) P \right], \end{aligned} \quad (1)$$

where we have adopted natural “transactional” units of time t , and defined the *Pareto potentials*,

$$A(w, t) := \frac{1}{N} \int_w^\infty dx P(x, t) \quad (2)$$

$$L(w, t) := \frac{1}{W} \int_0^w dx P(x, t) x \quad (3)$$

$$B(w, t) := \frac{1}{N} \int_0^w dx P(x, t) x^2 / 2, \quad (4)$$

and where in turn we have defined the total number of agents and the total wealth,

$$N := \int_0^\infty dx P(x, t) \quad (5)$$

$$W := \int_0^\infty dx P(x, t) x, \quad (6)$$

respectively. Equation (1) for the evolution of P is a nonlinear, integrodifferential Fokker-Planck equation. We note that there are only two free parameters in this equation, namely the redistribution coefficient τ and the WAA coefficient ζ .

If we are interested only in steady-state solutions, we set the time derivative to zero in Eq. (1), and integrate once with respect to w to obtain the nonlinear, integrodifferential, *ordinary* differential equation

$$\begin{aligned} \frac{d}{dw} \left[\left(B + \frac{w^2}{2} A \right) P \right] = & \left\{ \tau \left(\frac{W}{N} - w \right) \right. \\ & \left. - \zeta \left[2 \frac{N}{W} \left(B - \frac{w^2}{2} A \right) + (1 - 2L) w \right] \right\} P. \end{aligned} \quad (7)$$

Henceforth, we focus on fitting to solutions of this equation, and we ignore the time dependence, writing $P(w)$ instead of $P(w, t)$, etc.

The boundary conditions for Eq. (7) are described in [1]. We certainly expect

$$P(0) = \lim_{w \rightarrow \infty} P(w) = 0, \quad (8)$$

from which it immediately follows that

$$A(0) = 1 \quad \text{and} \quad L(0) = B(0) = 0. \quad (9)$$

From the normalization condition, Eq. (5), and the definition of A , Eq. (2), it also follows that

$$\lim_{w \rightarrow \infty} A(w) = 0. \quad (10)$$

At this point, we might wish to add that from the definition of W , Eq. (6), and the definition of L , Eq. (3), it also follows that $\lim_{w \rightarrow \infty} L(w) = 1$, but for the purposes of finding distributional solutions of Eq. (7), it is better not to assert this. Instead, Eqs. (8) through (10) alone are sufficient to make Eq. (7) well posed. Then, by either asymptotic analysis of the equation or numerical solutions thereof [1], we find that

$$\lim_{w \rightarrow \infty} L(w) = \begin{cases} 1 & \text{for } \zeta \leq \tau \\ \frac{\tau}{\zeta} & \text{for } \zeta > \tau. \end{cases} \quad (11)$$

The question that naturally arises is how Eq. (11) can be consistent with Eqs. (6) and (3). The answer is that for $\zeta > \tau$, the fraction of wealth $1 - \tau/\zeta$ “condenses” into the hands of a single agent – or, more precisely, a vanishingly small number of agents. When this “missing wealth” is included, the total wealth of the population is $(\tau/\zeta)W + (1 - \tau/\zeta)W = W$. This is described in much greater detail in [13].

III. THE LORENZ CURVE

Rather than plot $P(w)$ versus w , we instead plot the fraction of total wealth held by agents with wealth less than w , namely $L(w)$, versus the fraction of agents with wealth less than w , namely $F(w) := \frac{1}{N} \int_0^w dx P(x) = 1 - A(w)$. This is a parametric plot in the unit square, with parameter w , called a *Lorenz curve*. It passes through the origin $(0, 0)$, since $F(0) = L(0) = 0$. Below criticality, that is for $\zeta \leq \tau$, it passes through the point $(1, 1)$, since $\lim_{w \rightarrow \infty} F(w) = \lim_{w \rightarrow \infty} L(w) = 1$. Above criticality, that is for $\zeta > \tau$, it passes through $(1, \tau/\zeta)$ instead, because $\lim_{w \rightarrow \infty} L(w) = \tau/\zeta$. In between, it can be proven that it lies on or below the diagonal, is non-decreasing, and is concave up. (See, e.g., [7].)

Given N and W , the distribution P can be recovered from the Lorenz curve by noting that $F'(w) = P(w)/N$ and $L'(w) = P(w)w/W$. From this it follows that $dL/dF = \frac{w}{W/N}$, which is known as a function of F if the Lorenz curve is known. This may, in principle, be inverted to obtain F as a function of w , which may finally be differentiated to obtain $P(w) = NF'(w)$.

In the event that all agents have exactly the same wealth, the Lorenz curve is the diagonal. In the more realistic situation for which it lies below the diagonal, twice the area between the actual Lorenz curve and the diagonal is called the Gini coefficient,

G . Thus, $G = 0$ corresponds to complete economic equality, and $G = 1$ corresponds to complete wealth condensation. It is straightforward to show [7] that

$$G = 1 - \frac{2}{W} \int_0^\infty dw P(w) A(w) w. \quad (12)$$

In what follows, we shall compare the actual Lorenz curve of the U.S. in 2013 both without and with the constraint that the Gini coefficient match the actual wealth Gini coefficient for the U.S. at that time.

IV. DESCRIPTION OF DATA USED

The data we used for the U.S. wealth distribution was taken from the 2013 Survey of Consumer Finances (SCF) conducted by the Federal Reserve Board in cooperation with the U.S. Department of the Treasury [14]. It is a triennial cross-sectional survey of U.S. families, which includes information on families' balance sheets, pensions, income, and demographic characteristics.

Household wealth is represented in the SCF data by a variable called **networth**, which is an aggregate of several financial variables. Because **networth** includes both assets and debts, its value can be negative. This is a problem for comparisons with predictions of the YSM, which does not allow for negative w . Simply shifting all agents' wealth upward to make the data positive is not an option, because the agent furthest in debt actually owes billions of dollars. To solve this problem, we follow the approach of Norton and Ariely [15] who apparently set all observations with negative **networth** to zero, thereby effectively rendering all debtors bankrupt for the purposes of this analysis.

For reasons of confidentiality, the published SCF data [16] purposely does not include people listed on the famous "Forbes 400" list of the wealthiest people in the U.S. Because this group of people, though small in number, are so wealthy as to have a significant impact on macroeconomic indicators, we felt it important to add them back into the SCF data. Fortunately, the journal *Forbes* publishes this list of people on an annual basis, including values for their net wealth, so it was a simple matter to modify the SCF data to include them. Once this was done, we used Eq. (12) to calculate the wealth Gini coefficient of the U.S. in 2013, and found that it came to 0.8428.

V. FITTING THE DATA

Because the slope of the Lorenz curve ranges from zero to infinity, a simple least-squares fit of the solution of Eq. (7) to the data will be inaccurate. It is

preferable to measure the discrepancy by the sum of the Euclidean distances from the solution curve to the data points.

Suppose $L(F)$ is a Lorenz curve resulting from a solution to Eq. (7), and (f_i, l_i) the i th data point, as in Fig. 1. Suppose further that a perpendicular segment from the data point to the curve intersects it at $(\Phi_i, L(\Phi_i))$. A vector perpendicular to the curve at that point is $(L'(\Phi_i), -1)$, so we must have $(\Phi_i, L(\Phi_i)) + S_i(L'(\Phi_i), -1) = (f_i, l_i)$ for some S_i . Eliminating S_i from this vector equation yields

$$\Phi_i + (L(\Phi_i) - l_i) L'(\Phi_i) = f_i. \quad (13)$$

This can be numerically solved for Φ_i for each data point (f_i, l_i) . The total discrepancy is then given by

$$J := \sum_i \sqrt{(\Phi_i - f_i)^2 + (L(\Phi_i) - l_i)^2}. \quad (14)$$

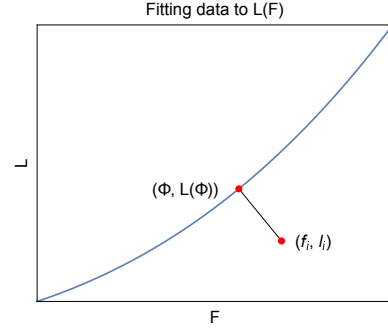


FIG. 1. Geometry of finding the point $(\Phi, L(\Phi))$ on the curve $L(F)$ that is closest to the external point (f_i, l_i) .

Because the function $L(F)$ describing the Lorenz curve of a solution to Eq. (7) ultimately depends only on τ and ζ , the values of Φ_i obtained by solving Eq. (13) will also depend on these two parameters, and hence so will the discrepancy given by Eq. (14), which, accordingly, we henceforth denote by $J(\tau, \zeta)$.

VI. RESULTS

We minimize $J(\tau, \zeta)$ in two ways. First, we search for its local minimum with respect to τ and ζ . Second, we employ Eq. (12) as a constraint, demanding that G of the fit match that of the data exactly. Clearly the first method will yield a lower discrepancy because it does not enforce the constraint.

Figure 2 displays $J(\tau, \zeta)$, plotted against τ and ζ . Its minimum, marked with a red dot, was found to occur at $\tau = 0.024$ and $\zeta = 0.028$, where $G = 0.8340$, close to but not precisely the actual value of 0.8428. Because $\zeta > \tau$ at this minimum, we see that

some wealth condensation was present in the U.S. in 2013. Specifically, in the continuum limit, a fraction $1 - \tau/\zeta \approx 0.14$ of U.S. wealth is in the hands of a “vanishingly small” fraction of agents.

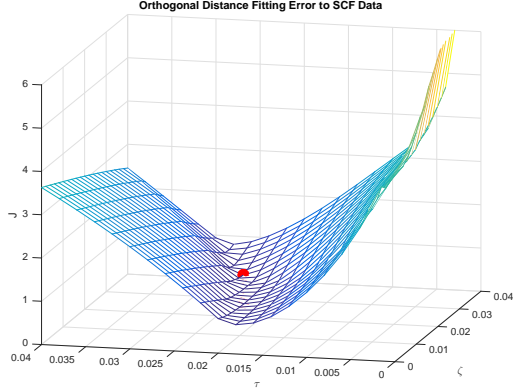


FIG. 2. Discrepancy $J(\tau, \zeta)$ plotted on (τ, ζ) plane, with minimum at $\tau = 0.024$ and $\zeta = 0.028$. Because $\zeta > \tau$, this falls into the coexistence regime of the EYSM, indicating partial wealth condensation in U.S. in 2013.

Figure 3 shows the Lorenz curve for the optimal values, $\tau = 0.024$ and $\zeta = 0.028$. From the picture, we see that the solution of Eq. (7) matches the actual Lorenz curve of the U.S. in 2013 extraordinarily well, suggesting that the EYSM captures important macroeconomic features of wealth distributions. The fit is noticeably better at medium and higher values of F than it is at lower values of F , probably because the YSM assumes positive wealth.

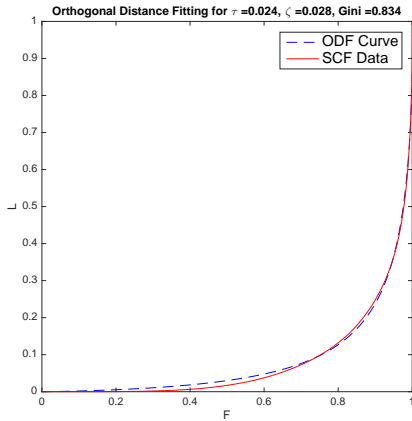


FIG. 3. Solution to Eq. (7) for $\tau = 0.024$ and $\zeta = 0.028$ (blue, dashed curve), and the actual Lorenz curve from the U.S. in 2013 (red curve).

Next, we employed Eq. (12) as a constraint, fix-

ing the Gini coefficient to be that of the data, namely 0.8428. This effectively eliminates a degree of freedom, reducing the two-parameter fit to a one-parameter fit. We searched along the (τ, ζ) pairs that yield the required Gini coefficient for the pair that minimized $J(\tau, \zeta)$. This is equivalent to finding the local minimum on the Gini coefficient level curve in Fig. 2. The result is displayed in Fig. 4, where the red surface is the constraint that $G = 0.8428$. Figure 5 shows more clearly how the discrepancy varies along this constraint curve. We see that, owing to the constraint, we can no longer reach the local minimum of $J(\tau, \zeta)$ precisely. Instead, we find a constrained local minimum at $\tau = 0.02117$ and $\zeta = 0.240$, still in the coexistence regime, now with a fraction $1 - \tau/\zeta \approx 0.12$ of the wealth in the condensed phase.

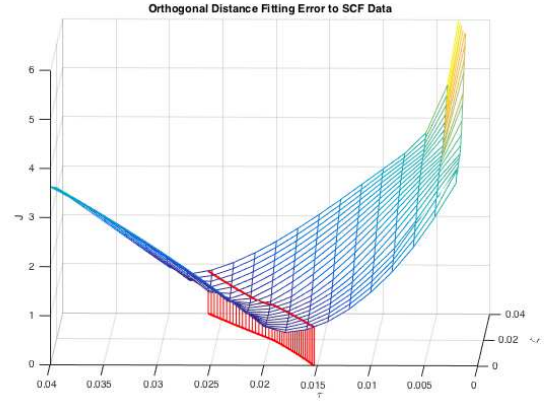


FIG. 4. Discrepancy $J(\tau, \zeta)$ plotted on (τ, ζ) plane. The red surface corresponds to the level curve $G = 0.8428$.

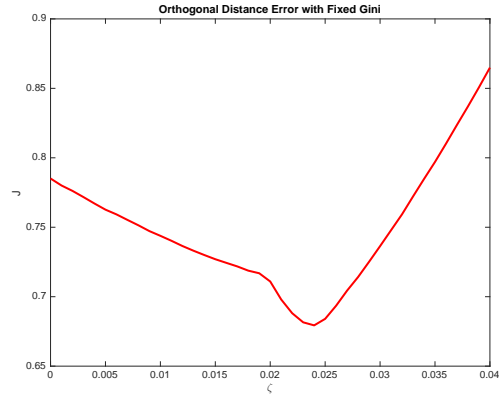


FIG. 5. Plot of $J(\tau, \zeta)$ on the level curve of Gini, as a function of ζ . This is essentially a magnified version of the red curve in Fig. 4

Finally, Fig. 6 shows the Lorenz curve fit for the

optimal values, $\tau = 0.02117$ and $\zeta = 0.0240$, subject to the $G = 0.8428$ constraint. The fit is still remarkably good, though not as good as the unconstrained fit. We can see that, for intermediate values of F , the fit curve lies below the true data to compensate the error incurred at lower values of F , in order to fix the Gini at the demanded value.

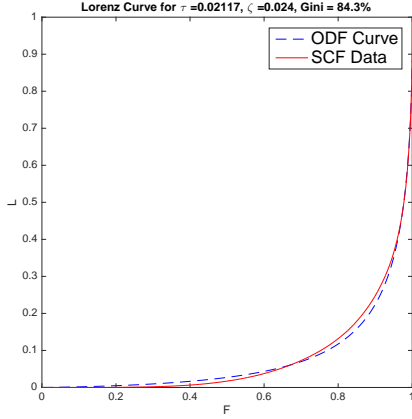


FIG. 6. Lorenz curve with $\tau = 0.02117$ and $\zeta = 0.024$, which minimizes the discrepancy $J(\tau, \zeta)$, subject to the constraint that $G = 0.8428$.

In the end, fitting without enforcing the constraint on the Gini coefficient may be the more useful of the two approaches. There is, after all, no reason to take the calculated Gini coefficient as a more important measure of the distribution than individual data points. Indeed, this appears to be the case when one compares the middle region of Fig. 3 to that of Fig. 6.

VII. CONCLUSIONS

Our data analysis and numerical studies demonstrate that the EYSM with only two free scalar parameters, τ representing redistribution and ζ representing WAA, is capable of describing the actual wealth distribution of the U.S. with remarkable accuracy. By fitting the two free parameters to the 2013 SCF data, we were able to estimate τ and ζ for the U.S. at that time. Because $\zeta > \tau$, we concluded that the U.S. economy was in the coexistence regime at that time – that is, a finite fraction of the wealth of the country was held by a vanishingly small fraction of economic agents.

Because we ignored the time dependency, and used only the steady state of Eq. (1), future work might focus on the inclusion of this term. For this purpose, a more sophisticated fit would be necessary, involving time as an independent variable, and using SCF data from prior years.

This work also indicates the most serious deficiency of the EYSM – namely, its inability to model the effect of agents with negative wealth. It is likely that this explains the inaccuracy of the Lorenz curves in both Figs. 3 and 6 at low values of F . A more accurate explanation of this regime, with the actual negative values of L , remains an important theoretical challenge.

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